Lecture 13

Einstein's field equations

Objectives:

- *The GR field equations*

Reading: *Schutz, 6; Hobson 7; Rindler 10.*

13.1 Symmetries of the curvature tensor

With 4 indices, the curvature tensor has a forbidding 256 components. Luckily several symmetries reduce these substantially. These are best seen in fully covariant form:

$$R_{\alpha\beta\gamma\delta} = g_{\alpha\rho} R^{\rho}{}_{\beta\gamma\delta},$$

for which symmetries such as

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta},$$

and

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma}.$$

can be proved. These relations reduce the number of independent components to 20.

Handout 6

These symmetries also mean that there is only one independent contraction

$$R_{\alpha\beta} = R^{\rho}{}_{\alpha\beta\rho},$$

because others are either zero, e.g.

$$R^{\rho}{}_{\rho\alpha\beta} = g^{\rho\sigma} R_{\sigma\rho\alpha\beta} = 0,$$

or the same to a factor of ± 1. $R_{\alpha\beta}$ is called the Ricci tensor, while its contraction

$$R = g^{\alpha\beta} R_{\alpha\beta},$$

is called the Ricci scalar.

NB Signs vary between books. I follow Hobson et al and Rindler.

13.2 The field equations

We seek a relativistic version of the Newtonian equation

$$\nabla^2\phi = 4\pi G\rho.$$

The relativistic analogue of the density ρ is the stress–energy tensor $T^{\alpha\beta}$.

ϕ is closely related to the metric, and ∇^2 suggests that we look for some tensor involving the second derivatives of the metric, $g_{\alpha\beta,\gamma\delta}$, which should be a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor like $T^{\alpha\beta}$.

The contravariant form of the Ricci tensor satisfies these conditions, suggesting the following:

$$R^{\alpha\beta} = kT^{\alpha\beta},$$

where k is some constant. (NB both $R^{\alpha\beta}$ and $T^{\alpha\beta}$ are symmetric.)

However, in SR $T^{\alpha\beta}$ satisfies the conservation equations ${T^{\alpha\beta}}_{,\alpha} = 0$ which in GR become

$${T^{\alpha\beta}}_{;\alpha} = 0,$$

whereas it turns out that

$${R^{\alpha\beta}}_{;\alpha} = \frac{1}{2}R_{,\alpha}g^{\alpha\beta} \neq 0,$$

where R is the Ricci scalar. Therefore $R^{\alpha\beta} = kT^{\alpha\beta}$ cannot be right. Handout 6

Fix by defining a new tensor, the Einstein tensor

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta},$$

because then

$${G^{\alpha\beta}}_{;\alpha} = \left(R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}\right)_{;\alpha} = {R^{\alpha\beta}}_{;\alpha} - \frac{1}{2}R_{;\alpha}g^{\alpha\beta} - \frac{1}{2}R{g^{\alpha\beta}}_{;\alpha} = 0,$$

since $\nabla\mathbf{g} = 0$ and $R_{;\alpha} = R_{,\alpha}$. Therefore we modify the equations to

$$\boxed{R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = kT^{\alpha\beta}.}$$

These are Einstein's field equations.

13.3 The Newtonian limit

The equations must reduce to $\nabla^2\phi = 4\pi G\rho$ in the case of slow motion in weak fields. To show this, it is easier to work with an alternate form: contracting

the field equations with $g_{\alpha\beta}$ then

$$g_{\alpha\beta}R^{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}g^{\alpha\beta} = kg_{\alpha\beta}T^{\alpha\beta},$$

and remembering the definition of R and defining $T = g_{\alpha\beta}T^{\alpha\beta}$,

$$R - \frac{1}{2}\delta^{\alpha}_{\alpha}R = -R = kT,$$

since $\delta^{\alpha}_{\alpha} = 4$. Therefore

$$\boxed{R^{\alpha\beta} = k\left(T^{\alpha\beta} - \frac{1}{2}Tg^{\alpha\beta}\right).}$$

Easier still is the covariant form:

$$R_{\alpha\beta} = k\left(T_{\alpha\beta} - \frac{1}{2}Tg_{\alpha\beta}\right).$$

The stress–energy tensor is

$$T_{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right)U_{\alpha}U_{\beta} - pg_{\alpha\beta}.$$

In the Newtonian case, $p/c^2 \ll \rho$, and so

$$T_{\alpha\beta} \approx \rho U_{\alpha}U_{\beta}.$$

Therefore

$$T = g^{\alpha\beta}T_{\alpha\beta} = \rho g^{\alpha\beta}U_{\alpha}U_{\beta} = \rho c^2.$$

Weak fields imply $g_{\alpha\beta} \approx \eta_{\alpha\beta}$, so $g_{00} \approx 1$. For slow motion, $U^i \ll U^0 \approx c$, and so $U_0 = g_{0\alpha}U^{\alpha} \approx g_{00}U^0 \approx c$ too. Thus

$$T_{00} \approx \rho c^2,$$

is the only significant component.

The 00 cpt of $R_{\alpha\beta}$ is:

$$R_{00} = \Gamma^{\rho}{}_{0\rho,0} - \Gamma^{\rho}{}_{00,\rho} + \Gamma^{\sigma}{}_{0\rho}\Gamma^{\rho}{}_{\sigma 0} - \Gamma^{\sigma}{}_{00}\Gamma^{\rho}{}_{\sigma\rho}.$$

All Γ are small, so the last two terms are negligible. Then assuming time-independence,

$$R_{00} \approx -\Gamma^{i}{}_{00,i}.$$

But, from the lecture on geodesics (chapter 11),

$$\Gamma^{i}{}_{00} = \frac{\phi_{,i}}{c^2}.$$

Thus

$$R_{00} \approx -\frac{1}{c^2}\phi_{,ii} = -\frac{1}{c^2}\frac{\partial^2 \phi}{\partial x^i \partial x^i} = -\frac{1}{c^2}\nabla^2\phi.$$

No longer balancing up/down indices since we are referring to spatial components only in nearly-flat space-time.

Finally, substituting in the field equations

$$-\frac{1}{c^2}\nabla^2\phi = k\left(\rho c^2 - \frac{1}{2}\rho c^2\right),$$

or

$$\nabla^2\phi = -\frac{kc^4}{2}\rho.$$

Therefore if $k = -8\pi G/c^4$, we get the Newtonian equation as required, and the field equations become

$$\boxed{R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta} = -\frac{8\pi G}{c^4}T^{\alpha\beta}.}$$

Key points:

- The field equations are second order, non-linear differential equations for the metric
- 10 independent equations replace $\nabla^2\phi = 4\pi G\rho$
- By design they satisfy the energy-momentum conservation relations $T^{\alpha\beta}{}_{;\alpha} = 0$
- The constant $8\pi G/c^4$ gives the correct Newtonian limit
- Although derived from strong theoretical arguments, like any physical theory, they can only be tested by <u>experiment</u>.